Beauty of Mathematics Pset #3 Solutions

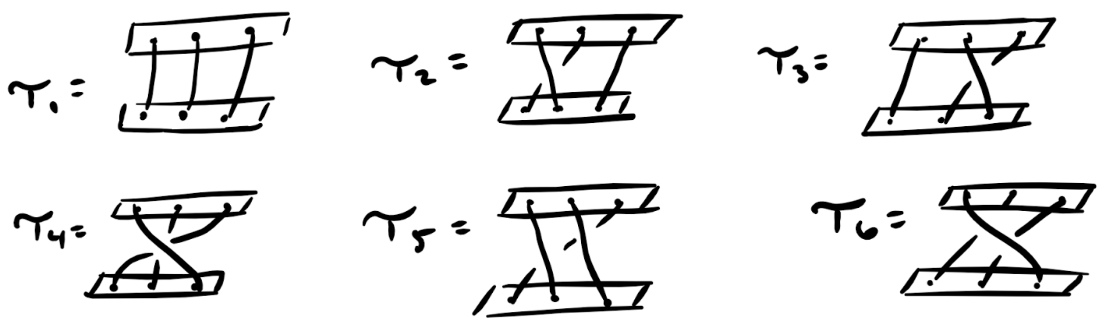
1. Write down all of the elements of the symmetric group on 3 letters. How many are there?

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\sigma_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad \sigma_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad \sigma_6 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

There are 6 elements. (The sigma_1 sigma_2 etc. naming is just for ease. If you wrote down these 6 elements, you're correct!)

2. Write down all of the elements of the braid group on 3 strands. How many are there?



Also 6 elements! (Again, the tau_1, tau_2, etc. is just naming the elements so it's easier to talk about them. :))

Notice:

There were a few questions about the Braid group: I wanted to write it here so everyone sees. For our purposes, it doesn't matter where the strands are in relationship to each other. That is, these two elements of the braid group are considered the same, even though the 3rd strand is "behind" the first two on the left, and "in front" of the first two on the right. So if your braid group looks slightly different than mine, not to worry! Just check that the strands are going in the right places. :)

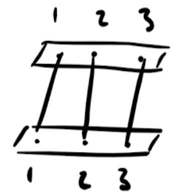
3. Can you give a matching between the two sets? If you can't, explain why. Is there anything similar about these two groups?

Mapping:

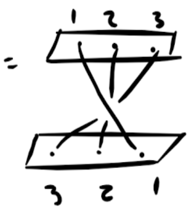
$$\sigma_1 \longmapsto \tau_1$$
$$\sigma_2 \longmapsto \tau_2$$
$$\sigma_3 \longmapsto \tau_3$$
$$\sigma_4 \longmapsto \tau_4$$
$$\sigma_5 \longmapsto \tau_5$$
$$\sigma_6 \longmapsto \tau_6$$

Pictorally:

ex 1: $\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \longmapsto \tau_1 =$



ex 2: $\sigma_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \longmapsto \tau_4 =$



(The special arrow is the matching/mapping. It says that specific element gets matched with/mapped to the corresponding specific element.)
We know there is a matching between these two sets because they both have six elements. But there's something special about this matching! This matching is between *groups*, and it's said to "preserve structure". We can see that the identity of the symmetric group (sigma 1) gets matched with the identity of the braid group (tau 1). And sigma 4 looks a lot like tau 4, when you label the strands of the braid groups.

This special kind of matching between the two groups is called an *isomorphism*. It means that even though the groups have entirely different elements in them and the binary operation is different, they still *look* and *act* the same way! While this example may have been hard to grasp, I think it beautifully illustrates this idea of similarity in mathematics. We have two objects that look entirely different, but act the same way. Because found this special kind of relationship, if we discover something new about the symmetric group, we also discovered it for the braid group!

There are all different kinds of relationships and matchings between sets, groups, and other mathematical objects. A lot of mathematics seems to be how things relate to one another and what kind of information you can extract from those relationships!

4. Prove that the set of rotations on a square is a group. That is, show:
    i. The set contains an identity element (a "do nothing" element).
    ii. Every element of the group has an inverse (an element that reverses each rotation)
    iii. Show that composing rotations is associative.
    iv. Show that the set is closed under the operation of rotation. (For any rotation you do, you'll still have a square.)

Firstly, someone was kind enough to point out that iv. is incorrectly worded! It should say that for any two rotations you do, you get a third rotation! Because the set is not a set of squares, it is a set of rotations. So *the set of rotations of closed under composition.*
Let's show:

i. The set contains an identity element: the identity element is the 0 degree rotation.

ii. Every element of the group has an inverse: the inverse of any rotation of x degrees is the rotation of -x degrees! You end up with the same thing that you started with, which is what you want.

iii. Composing rotations is the same thing as adding their degrees together. So doing a 20 degree rotation, followed by a 30 degree rotation, followed by a 40 degree rotation is the same as doing a 20 + 30 + 40 degree rotation. And 20 + 30 + 40 = (20 + 30) + 40 = 20 + (30 + 40) = 90. So we're golden!

You'll notice this is not a careful proof of associativity, but merely an example. This example can, however, be generalized to give a full and abstract proof if needed.

iv. Lastly, that the set of closed under the operation of *composition.* Any two rotations of x and y degrees will give you a third rotation of (x + y) degrees. :)

Feel free to email if you have questions!